Counting

Methods

What is the **Fundamental Counting Rule**?

If event A can occur in **m** ways and event B can occur in **n** ways, then the number of ways that event A happens first and then event B happens is given by **m • n**.

Example:

How many ways can we select a card from a full deck of playing cards followed by selecting a number from $1, 2, 3, \cdots, 20$?

Solution:

There are 52 cards in a full deck of playing cards and there are 20 numbers to choose from the list $1, 2, 3, \cdots, 20$, therefore

$$\begin{aligned} m \bullet n &= 52 \bullet 20 \\ &= 1040 \text{ ways} \end{aligned}$$

Example:

Find the number of different lock combinations for the bike lock shown below if the same digit can be chosen repeatedly.

Solution:

Since there are 10 choices for each digit, and we have to select 4 digits that can be chosen repeatedly, therefore

$$10 \bullet 10 \bullet 10 \bullet 10 \quad = \quad 10,000 \text{ ways}$$

Example:

How many four-digit numbers can be formed from the digits 1, 2, 4, 6, 7, and 9 if each digit can be used only once?

Solution:

The first digit can be chosen from any of the 6 digits, and cannot be used again.

Now we have 5 digits to choose from for the second digit and this selection cannot be used again.

Therefore by using the fundamental counting rule, we get

$$6 \cdot 5 \cdot 4 \cdot 3 = 360 \text{ different four-digit numbers}$$

What is $n!$?

The symbol ! is called factorial and $n!$ is the product of decreasing natural numbers starting with n.

$$\begin{aligned} n! &= n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1, \text{ and by definition} \\ 0! &= 1 \end{aligned}$$

Example:

Find 6!.

Solution:

Using the definition of factorial, we get

$$\begin{aligned} 6! &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 720 \end{aligned}$$

Example:

Simplify $\dfrac{10!}{2! \bullet 8!}$.

Solution:

Using the definition of factorial, we get

$$\frac{10!}{2! \bullet 8!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdots 3 \cdot 2 \cdot 1}{2 \cdot 1 \bullet 8 \cdot 7 \cdots 3 \cdot 2 \cdot 1} = 45$$

Example:

Simplify $\dfrac{12!}{3! \bullet 4! \bullet 5!}$.

Solution:

Using the definition of factorial, we get

$$\frac{12!}{3! \bullet 4! \bullet 5!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \bullet 4 \cdot 3 \cdot 2 \cdot 1 \bullet 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 27720$$

What is a **Permutation**?

It is the study of all possible rearranging or reordering a list of n objects.

What is the purpose of studying **Permutation**?

- To compute the number of ways of all possible arrangements.
- To compute probabilities of certain arrangement.

What is the **Factorial Rule**?

It is the way to compute the number of all possible arrangements when

- There are n items, and they are all different.
- We select all of the n items without replacement.

If these requirements are satisfied, then the number of permutations of all items is $n!$.

Example:

How many five-letter words can we make by using the letters A, B, C, D, and E?

Solution:

Using the factorial rule, the answer is $5! = 120$ different words.

Example:

Lisa has four different books and wishes to place them in the bookshelf. Find the number of ways that she can arrange these books in the bookshelf.

Solution:

She has 4 possible selections for the first book to be placed in the bookshelf, then she has 3 possible selections to choose the second book, and so on with only 1 book left for the last selection. So by using the **fundamental counting rule**, or **factorial rule**. We get

$$\begin{aligned} 4 \cdot 3 \cdot 2 \cdot 1 &= 4! \\ &= 24 \text{ ways to arrange 4 books in the bookshelf.} \end{aligned}$$

What is the **Modified Factorial Rule**?

It is the way to compute the number of all possible arrangements when

- There are n items, and some items are identical to others.
- We select all of the n items without replacement.
- Rearrangements of distinct items are considered different arrangements

If these requirements are satisfied, then the number of permutations of all items is given by $\dfrac{n!}{n_1! \cdot n_2! \cdot n_3! \cdots n_k!}$ where n_1 alike, n_2 alike, and so on with $n_1 + n_2 + \cdots n_k = n$.

Example:

Using the letters in the word **BALLOON** , how many distinct 7-letter words can we make?

Solution:

Since there are seven letters in the balloon, so $n = 7$, and the letters B, A, and N all appear once while letters L, and O each appear twice. Now using modified factorial rule, we get

$$\frac{7!}{1! \cdot 1! \cdot 2! \cdot 2! \cdot 1!} = \text{1260 different words}$$

What is ${}_nP_r$?

It is the way to compute the number of all possible arrangements when

- There are n items, and they are all different.
- We select r items without replacement.
- Order of arrangements matter.

If these requirements are satisfied, then the number of

permutations is given by

$$\mathbf{{}_nP_r = \frac{n!}{(n-r)!}}$$

Example:

In a recent city election, all registered voters were supposed to vote for mayor, a controller and a city attorney. There were 8 candidates running for these 3 positions. How many ways can this be done?

Solution:

Since there are eight candidates, so $n = 8$, and the order of the selection matters by positions, so the number of ways this can be done is given by

$$\begin{aligned} {}_8P_3 &= \frac{8!}{(8-3)!} \\ &= 336 \text{ different ways} \end{aligned}$$

What is ${}_nC_r$?

It is the way to compute the number of all possible arrangements when

- There are n items, and they are all different.
- We select r items without replacement.
- Order of arrangements does not matter.

If these requirements are satisfied, then the number of **combinations** is given by

$$\mathbf{{}_nC_r = \frac{n!}{r! \cdot (n-r)!}}$$

Example:

In a recent city election, all registered voters were supposed to vote to select 3 city counsel members. There were 7 male, 3 female candidates that were running for these 3 positions.

- How many ways can we select three candidates?
- How many ways can we select 2 males and 1 female?

Solution:

Since there are 10 candidates, and we need to select three city counsel members, furthermore the order of the selection does not matters, so the number of ways this can be done is given by

$$\begin{aligned} {}_{10}C_3 &= \frac{10!}{3! \cdot (10-3)!} \\ &= 120 \text{ different ways} \end{aligned}$$

Solution Continued:

The number of ways for selecting 2 males and 1 female, can be computed by

$$\begin{aligned} {}_7C_2 \cdot {}_3C_1 &= 21 \cdot 3 \\ &= 63 \text{ different ways} \end{aligned}$$

Now if we wish to find the number of ways of selecting 1 male and 2 females, we can compute that by

$$\begin{aligned} {}_7C_1 \cdot {}_3C_2 &= 7 \cdot 3 \\ &= 21 \text{ different ways} \end{aligned}$$
